

Invariant amplitudes for coherent electromagnetic pseudoscalar production from a spin-one target *

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Abstract

A set of 13 linearly independent invariant amplitudes for the electromagnetic production of a pseudoscalar particle from a spin-one particle is derived which respect Lorentz and gauge invariance. The T -matrix can be represented by a linear superposition of these amplitudes with invariant functions as coefficients, which depend on the Mandelstam variables only. Their explicit form is determined by the underlying dynamics. Nine of these amplitudes are purely transverse and describe photoproduction. The remaining four appear in electroproduction in addition, describing charge and longitudinal current contributions. Furthermore, a reduction of these amplitudes to operators acting in non-relativistic spin space is given.

1 Introduction

In recent years, coherent pseudoscalar meson production in electromagnetic reactions on nuclei has become a very active field of research in medium energy physics. A particularly interesting role plays the deuteron, since it is the simplest nucleus on whose structure we have abundant information and a reliable theoretical understanding. Moreover, it constitutes the simplest and cleanest neutron target, and thus allows one to study neutron properties, provided one has control on binding effects in the most general sense. In fact, this is one main motivation for studying coherent pion and eta photo- and electroproduction on the deuteron, both in experiment and theory (see, for example [1, 2, 3, 4] and references therein). Most theoretical approaches thus start from the one-body contributions to the reaction matrix which, therefore, is given by specific contributions in terms of the nucleon variables. However, it might be useful to investigate the general structure of the reaction amplitude without any specific input from the internal degrees of freedom of the target, using only the general principles of basic conservation laws, like Lorentz covariance, gauge invariance etc. In other words, we want to determine the most general framework for the formal description of the reaction matrix into which any specific reaction model has to fit. This is in analogy to the CGLN-amplitudes for pion photo- and electroproduction on a nucleon [5, 6].

To this end, we first construct in Sect. 2 the most general form of the transition amplitude in terms of basic amplitudes which respect Lorentz covariance and parity conservation. Then, in Sect. 3 we determine from these basic amplitudes a set of 13 linearly independent amplitudes which in addition respect gauge invariance. Helicity amplitudes are derived in Sect. 4. They serve as a very convenient basis for the construction of observables. Finally, in the last section we relate these amplitudes to a corresponding basic set of 13 nonrelativistic operators acting in spin-one spinor space.

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2 Formal developments

The basic reaction, coherent pseudoscalar production by photoabsorption or electron scattering from a spin-one target, is of the form

$$\gamma^{(*)}(k) + a(p) \rightarrow a(p') + b(q), \quad (1)$$

where the real or virtual photon has momentum k , a is a spin-one particle of mass M with initial and final momenta p and p' , respectively, and b a pseudoscalar meson of mass m with momentum q . In the c.m. frame of photon and target particle, the differential cross section for electromagnetic production with real or virtual photons is in general given by

$$d\sigma_{fi}^{c.m.} = c_\gamma \text{tr}(\mathcal{M}_{fi}^\dagger \mathcal{M}_{fi} \hat{\rho}^\gamma \hat{\rho}^{target}) d\Omega_q^{c.m.}, \quad (2)$$

where $\hat{\rho}^\gamma$ and $\hat{\rho}^{target}$ denote the density matrices of the real or virtual photon and the target particle, respectively, and c_γ contains corresponding kinematic factors whose specific forms are not needed for the present study. The trace is to be taken with respect to all polarization quantum numbers.

\mathcal{M}_{fi} denotes the invariant matrix element which has to be linear in the polarization vectors of the participating particles. Therefore, it will have the general structure

$$\mathcal{M}_{fi} = U^{\mu\dagger}(\vec{p}', m') O_{\mu\nu\rho}(p', q, p, k) \epsilon^\nu(k, \lambda) U^\rho(\vec{p}, m), \quad (3)$$

where the initial and final polarization Lorentz vectors of the spin-one particle with spin projections m and m' are denoted by $U^\rho(\vec{p}, m)$ and $U^\mu(\vec{p}', m')$, respectively. The explicit form of the polarization vector $U(\vec{p}, m)$ of a massive spin-one particle is given by

$$U^\mu(\vec{p}, m) = \frac{1}{M} (\vec{p} \cdot \vec{e}_m, M \vec{e}_m + \frac{\vec{p} \cdot \vec{e}_m}{E_p + M} \vec{p}), \quad (4)$$

where $E_p = \sqrt{\vec{p}^2 + M^2}$, and the rest frame polarization three-vector is denoted by \vec{e}_m . $U(\vec{p}, m)$ obeys the transversality condition

$$p_\rho U^\rho(\vec{p}, m) = 0. \quad (5)$$

The photon polarization vectors are denoted by $\epsilon^\nu(k, \lambda)$. For real photons one has only two transverse polarization states with helicity $\lambda = \pm 1$ which have only spatial components

$$\epsilon^\nu(k, \lambda = \pm 1) = (0, \vec{\epsilon}_{\pm 1}), \quad (6)$$

with $\vec{\epsilon}_{\pm 1} \cdot \vec{k} = 0$ and $|\vec{\epsilon}_{\pm 1}| = 1$, whereas for electroproduction in addition a third polarization state ($\lambda = 0$) appears having both, longitudinal and timelike components according to

$$\epsilon^\nu(k, \lambda = 0) = \frac{|\vec{k}|}{\sqrt{K^2}} (1, \frac{k_0}{|\vec{k}|^2} \vec{k}), \quad (7)$$

where we have introduced for convenience $K^2 = -k_\mu^2 > 0$. In both cases, one has the transversality condition

$$k_\nu \epsilon^\nu(k, \lambda) = 0. \quad (8)$$

Furthermore, assuming parity conservation, $O_{\mu\nu\rho}$ has to be a third rank Lorentz pseudotensor since a pseudoscalar is produced. Folded between the target polarization Lorentz vectors, it describes the axial current matrix elements for the coherent pseudoscalar production

$$j_{fi}^\nu(p', q, p, k) = U_\mu^\dagger(\vec{p}', m') O^{\mu\nu\rho}(p', q, p, k) U_\rho(\vec{p}, m). \quad (9)$$

As constraint from gauge invariance one has the condition of current conservation

$$k_\nu j_{fi}^\nu(p', q, p, k) = U_\mu^\dagger(\vec{p}', m') O^{\mu\nu\rho}(p', q, p, k) k_\nu U_\rho(\vec{p}, m) = 0. \quad (10)$$

The operator $O_{\mu\nu\rho}$ has to be constructed from the momenta of the participating particles, from the metric tensor $g^{\mu\nu}$ and the four-dimensional completely antisymmetric Levy-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$. The presence of the latter is necessary in order to guarantee the pseudotensor property. We will now

construct a set of basic covariant amplitudes $\{\Omega_\alpha\}$ which serves as a basis for the representation of the invariant matrix element as a linear superposition of the basic amplitudes, i.e.,

$$\mathcal{M}_{fi} = \sum_{\alpha} F_{\alpha}(s, t, u) \Omega_{\alpha}, \quad (11)$$

where $F_{\alpha}(s, t, u)$ denote invariant functions which depend solely on the Mandelstam variables s, t and u , defined as usual by

$$s = (k + p)^2, \quad t = (k - q)^2, \quad u = (k - p')^2. \quad (12)$$

Because of the condition

$$s + t + u = 2M^2 - K^2 + m^2, \quad (13)$$

only two of them are independent, for example, s and t . The specific form of the invariant functions $F_{\alpha}(s, t, u)$ will depend on the detailed dynamical properties of the target and the produced meson. Because of momentum conservation

$$k + p = p' + q, \quad (14)$$

only three of the particle momenta can be considered as independent. We will choose in the following k, p and q .

Counting the number of polarization states for the initial and final particles, one finds for electroproduction a total number of T -matrix elements of $3 \times 3 \times 3 = 27$ and for photoproduction 18, which reduces to 13, respective 9 independent T -matrix elements, if parity conservation can be invoked (see for example [2]). Consequently, one has to find 13 respective 9 linearly independent invariant amplitudes. By this we mean that a set $\{\Omega_{\alpha}\}$ is called linearly independent if one cannot find a set of scalar functions $\{f_{\alpha}(s, t, u)\}$ such that the equation

$$\sum_{\alpha} f_{\alpha}(s, t, u) \Omega_{\alpha} \equiv 0 \quad (15)$$

holds identically.

First we will select all possible amplitudes for $O^{\mu\nu\rho}$ respecting the requirements of Lorentz covariance and parity conservation. In view of the pseudotensor character, each amplitude has to be linear in the Levy-Civita tensor, because higher order odd powers of $\varepsilon^{\mu\nu\rho\sigma}$ can be reduced to linear expressions in $\varepsilon^{\mu\nu\rho\sigma}$ (see Appendix A). Any two amplitudes, which only differ by scalar products between the vectors k, p and q , cannot be considered as independent, because the scalar products can be expressed in terms of the Mandelstam variables according to

$$k \cdot p = \frac{1}{2} (s - M^2 + K^2), \quad (16)$$

$$k \cdot q = \frac{1}{2} (m^2 - K^2 - t), \quad (17)$$

$$p \cdot q = \frac{1}{2} (s + t - M^2 + K^2). \quad (18)$$

For the further discussion it is useful to introduce for convenience a covariant pseudoscalar by contraction of $\varepsilon^{\mu\nu\rho\sigma}$ with four Lorentz vectors a, b, c and d

$$S(a, b, c, d) = \varepsilon^{\mu\nu\rho\sigma} a_{\mu} b_{\nu} c_{\rho} d_{\sigma}. \quad (19)$$

Any amplitude thus has to contain such a pseudoscalar, and one can distinguish the possible types according to whether one, two or three of the polarization vectors are contained in $\varepsilon^{\mu\nu\rho\sigma}$. In other words, the possible candidates for $O^{\mu\nu\rho}$ are

$$\begin{aligned} & \varepsilon^{\mu\nu\rho\sigma} x_{\sigma}, \\ & \varepsilon^{\mu\nu\sigma\tau} x_{\rho} y_{\sigma} z_{\tau}, \quad \varepsilon^{\mu\rho\sigma\tau} x_{\nu} y_{\sigma} z_{\tau}, \quad \varepsilon^{\nu\rho\sigma\tau} x_{\mu} y_{\sigma} z_{\tau}, \\ & \varepsilon^{\mu\lambda\sigma\tau} g_{\nu\rho} z_{\lambda} u_{\sigma} v_{\tau}, \quad \varepsilon^{\nu\lambda\sigma\tau} g_{\mu\rho} z_{\lambda} u_{\sigma} v_{\tau}, \quad \varepsilon^{\rho\lambda\sigma\tau} g_{\mu\nu} z_{\lambda} u_{\sigma} v_{\tau}, \\ & \varepsilon^{\mu\lambda\sigma\tau} x_{\nu} y_{\rho} z_{\lambda} u_{\sigma} v_{\tau}, \quad \varepsilon^{\nu\lambda\sigma\tau} x_{\mu} y_{\rho} z_{\lambda} u_{\sigma} v_{\tau}, \quad \varepsilon^{\rho\lambda\sigma\tau} y_{\mu} y_{\nu} z_{\lambda} u_{\sigma} v_{\tau}, \end{aligned}$$

where $x, y, z, u, v \in \{k, p, q\}$. Since only three independent kinematic vectors are available, no pseudoscalar of the type $S(x, y, z, u)$ will appear. Therefore, one finds as basic types of amplitudes the ones listed in Table 1, where we use as a shorthand U' for $U^\dagger(\vec{p}', m')$ and U for $U(\vec{p}, m)$. Note, that in all following expressions the polarization vector U' of the final target state always has to stand on the left hand side with respect to U . Taking into account the fact, that $S(a, b, c, d)$ vanishes if two arguments are equal, one finds three different amplitudes for each type Ω_a and Ω_e , while for each of the others one finds nine. Thus the total number of possible amplitudes is 60, too many according to what has been said above about the number of independent amplitudes.

3 Construction of independent invariant amplitudes

The main task now is to determine from the set of amplitudes in Table 1 the independent ones. A first reduction is achieved by taking into account the transversality conditions (5) and (8) from which follows

$$\Omega_b(x, y, p) = 0, \quad (20)$$

$$\Omega_d(x, y, k) = 0, \quad (21)$$

$$\Omega_f(x, p) = 0, \quad (22)$$

$$\Omega_f(k, y) = 0, \quad (23)$$

$$\Omega_g(x, k) = 0, \quad (24)$$

$$\Omega_h(x, p) = 0, \quad (25)$$

and, because of $U' \cdot p = U' \cdot q - U' \cdot k$, one furthermore has

$$\Omega_c(x, y, p) = \Omega_c(x, y, q) - \Omega_c(x, y, k), \quad (26)$$

$$\Omega_{g/h}(p, y) = \Omega_{g/h}(q, y) - \Omega_{g/h}(k, y). \quad (27)$$

This leaves 36 amplitudes from which we have to find the independent ones which in addition respect the gauge condition of (10).

A further reduction follows from two linear relations for the Levy-Civita tensor (see the Appendix A). As is shown in detail in Appendix B, by means of these relations one can eliminate the amplitudes $\Omega_a(k)$ and $\Omega_{b/c}(p, q, q)$ (see the relations (B.31), (B.32) and (B.35) of Appendix B). Furthermore, one can eliminate $\Omega_{b/c}(p, q, k)$, Ω_d , Ω_f , and Ω_g according to the relations

$$\Omega_b(p, q, k) = -\Omega_b(k, p, q) - \Omega_e(U', \epsilon, U) + \Omega_e(\epsilon, U', U), \quad (28)$$

$$\Omega_c(p, q, k) = -\Omega_c(k, p, q) - \Omega_c(q, k, p) - \Omega_e(U, U', \epsilon) + \Omega_e(\epsilon, U', U), \quad (29)$$

$$\Omega_d(x, y, z) = \Omega_a(x) y \cdot z - \Omega_a(y) x \cdot z + \Omega_b(x, y, z) + \Omega_c(x, y, z), \quad (30)$$

$$\Omega_f(x, y) = \Omega_h(x, y) + Q_b(k, p, q; x, y), \quad (31)$$

$$\Omega_g(x, y) = \Omega_h(x, y) - Q_c(k, p, q; y, x), \quad (32)$$

where $Q_{b/c}$ is defined in (B.8) of Appendix B.

Of the remaining 17 linearly independent amplitudes, 13 ones are gauge invariant. They are listed in Table 2. It means that these amplitudes vanish under the replacement $\epsilon \rightarrow k$. The other four non-gauge invariant amplitudes are listed in Table 3. They cannot be linearly combined to form gauge invariant amplitudes and, therefore, cannot contribute to \mathcal{M}_{fi} . In view of the above mentioned 13 independent T -matrix elements, it is not surprising that we find exactly 13 linearly independent invariant amplitudes which respect gauge invariance and which could serve as a complete basis for \mathcal{M}_{fi} . However, not all of them have the most convenient form. While the first nine amplitudes of Table 2 are purely transverse in the c.m. frame of photon and target particle, and, therefore, suffice and are well suited for describing photoproduction, the remaining four, which in addition are needed in electroproduction, have besides charge and longitudinal current components also transverse current pieces. For this reason it is more advantageous to replace the last four amplitudes of Table 2 by equivalent amplitudes which are purely longitudinal in the c.m. frame. This is easily achieved by using the relations in (31) and (32) in conjunction with (23) and (24), yielding

$$\begin{aligned} k \cdot p \Omega_f(k, k) - k_\mu^2 \Omega_f(p, k) &= (k_\mu^2 + k \cdot p \Omega_h(k, k) - k_\mu^2 \Omega_h(q, k) \\ &\quad - [k, p; k, q] \Omega_b(k, p, k) + [k, p; k, p] \Omega_b(k, q, k), \end{aligned} \quad (33)$$

$$\begin{aligned}
k \cdot p \Omega_f(k, q) - k_\mu^2 \Omega_f(p, q) &= (k_\mu^2 + k \cdot p \Omega_h(k, q) - k_\mu^2 \Omega_h(q, q) \\
&\quad - [k, p; k, q] \Omega_b(k, p, q) + [k, p; k, p] \Omega_b(k, q, q) \\
&\quad + 2 [k, p; k, q] \Omega_e(\epsilon, U', U), \tag{34}
\end{aligned}$$

$$\begin{aligned}
k \cdot p \Omega_g(k, k) - k_\mu^2 \Omega_g(k, p) &= (k_\mu^2 + k \cdot p \Omega_h(k, k) - k_\mu^2 \Omega_h(k, q) \\
&\quad + [k, p; k, q] \Omega_c(k, p, k) - [k, p; k, p] \Omega_c(k, q, k), \tag{35}
\end{aligned}$$

$$\begin{aligned}
k \cdot p \Omega_g(q, k) - k_\mu^2 \Omega_g(q, p) &= (k_\mu^2 + k \cdot p \Omega_h(q, k) - k_\mu^2 \Omega_h(q, q) \\
&\quad + [k, p; k, q] \Omega_c(k, p, q) - [k, p; k, p] \Omega_c(k, q, q), \tag{36}
\end{aligned}$$

where the symbol $[a, b; c, d]$ is defined in (A.8). Therefore, we can replace the last four amplitudes of Table 2 by the equivalent ones as given on the left hand side of (33) through (36) and thus use the operators in Table 4 for the longitudinal contributions in electroproduction.

4 Helicity amplitudes

The most convenient framework for the description of observables like differential cross section, beam and target asymmetries and recoil polarization, is given by the helicity amplitudes in the c.m. frame of the reaction. For the coordinate system we choose the z -axis along the photon momentum \vec{k} , the y -axis perpendicular to the plane built by photon and meson momentum in the direction $\vec{k} \times \vec{q}$ and finally the x -axis along $\vec{e}_y \times \vec{e}_z$. The corresponding spherical basis will be denoted by \vec{e}_λ with $\lambda = 0, \pm 1$. The helicity states of the photon polarization are already given in (6) and (7). Similarly, the helicity states of the initial and final deuteron polarization are given by

$$U^\mu(\vec{p}, \lambda) = \frac{(-)^\lambda}{M} \left(k \delta_{\lambda 0}, -E(\lambda) \vec{e}_{-\lambda} \right), \tag{37}$$

$$U^\mu(\vec{p}', \lambda') = \frac{(-)^{\lambda'}}{M} \left(q \delta_{\lambda' 0}, -E'(\lambda') \sum_{\bar{\lambda}} \vec{e}_{\bar{\lambda}} d_{\bar{\lambda}, -\lambda}^1(\theta) \right), \tag{38}$$

where θ denotes the angle between \vec{k} and \vec{q} . Use has been made of the c.m. frame relations

$$p^\mu = (E_k, -k \vec{e}_0), \quad p'^\mu = \left(E_q, -q \sum_{\bar{\lambda}} \vec{e}_{\bar{\lambda}} d_{\bar{\lambda} 0}^1(\theta) \right). \tag{39}$$

In the c.m. frame the absolute values of the momenta $k = |\vec{k}|$ and $q = |\vec{q}|$ are given by the well-known expressions

$$\begin{aligned}
k^2 &= \frac{1}{4s} \left((\sqrt{s} + M)^2 + K^2 \right) \left((\sqrt{s} - M)^2 + K^2 \right) \\
&= \frac{1}{4s} \left((s - M^2 + K^2)^2 + 4K^2 M^2 \right), \tag{40}
\end{aligned}$$

$$\begin{aligned}
q^2 &= \frac{1}{4s} \left((\sqrt{s} + M)^2 - m^2 \right) \left((\sqrt{s} - M)^2 - m^2 \right) \\
&= \frac{1}{4s} \left((s - M^2 - m^2)^2 - 4m^2 M^2 \right). \tag{41}
\end{aligned}$$

The rotation matrices $d_{m'm}^j$ are in the convention of [7]. Furthermore we have introduced in the above expressions the shorthand notation

$$E(\lambda) = M \delta_{|\lambda| 1} + E_k \delta_{\lambda 0}, \tag{42}$$

$$E'(\lambda') = M \delta_{|\lambda'| 1} + E_q \delta_{\lambda' 0}. \tag{43}$$

Explicit evaluation of the various terms in the expressions for the invariant transverse and longitudinal amplitudes listed in Tables 2 and 4 yields

$$U'_{\lambda'} \cdot U_\lambda = \frac{1}{M^2} \left(k q \delta_{\lambda' 0} \delta_{\lambda 0} - E'(\lambda') E(\lambda) d_{\lambda \lambda'}^1(\theta) \right), \tag{44}$$

$$U_\lambda \cdot k = \frac{k}{M} \sqrt{s} \delta_{\lambda 0}, \tag{45}$$

$$U_\lambda \cdot q = \frac{1}{M} \left(k q_0 \delta_{\lambda 0} + q E(\lambda) d_{\lambda 0}^1(\theta) \right), \tag{46}$$

$$U'_{\lambda'} \cdot k = \frac{1}{M} \left(q k_0 \delta_{\lambda'0} + k E'(\lambda') d_{0\lambda'}^1(\theta) \right), \quad (47)$$

$$U'_{\lambda'} \cdot q = \frac{1}{M} q \sqrt{s} \delta_{\lambda'0}, \quad (48)$$

$$S(\epsilon_{\lambda\gamma}, k, p, q) = -i \lambda_\gamma k q \sqrt{s} d_{\lambda\gamma 0}^1(\theta), \quad (49)$$

$$S(U_\lambda, k, p, q) = -i \lambda k q \sqrt{s} d_{\lambda 0}^1(\theta), \quad (50)$$

$$S(U'_{\lambda'}, k, p, q) = i \lambda' k q \sqrt{s} d_{\lambda' 0}^1(\theta), \quad (51)$$

$$S(\epsilon_{\lambda\gamma}, U_\lambda, k, p) = \frac{i}{M} \lambda E(\lambda) k \sqrt{s} \delta_{\lambda\gamma\lambda}, \quad (52)$$

$$S(U'_{\lambda'}, \epsilon_{\lambda\gamma}, k, p) = \frac{i}{M} \lambda_\gamma E'(\lambda') k \sqrt{s} d_{-\lambda'\lambda\gamma}^1(\theta), \quad (53)$$

$$k \cdot p \, k \cdot \epsilon_{\lambda\gamma} - k \cdot k \, p \cdot \epsilon_{\lambda\gamma} = \sqrt{K^2} k \sqrt{s} \delta_{\lambda\gamma 0}. \quad (54)$$

Note that on the rhs we have denoted the absolute values of the photon and meson three momenta by k respective q . From these expressions it is straightforward to construct the helicity representation of the invariant amplitudes Ω_α . They are listed in Table 5. With the help of these expressions the helicity representation of the T -matrix is obtained easily in terms of the invariant functions $F_\alpha(s, t, u)$ according to (11). Note that the helicity amplitudes and thus the full T -matrix obey the symmetry relation which reflects parity conservation

$$\Omega_{\alpha, -\lambda' - \lambda_\gamma - \lambda} = (-)^{1+\lambda' + \lambda_\gamma + \lambda} \Omega_{\alpha, \lambda' \lambda_\gamma \lambda}, \quad (55)$$

as is evident from the explicit expressions in Table 5.

5 Representation with respect to nonrelativistic spin-one spinors

For the description of \mathcal{M}_{fi} in the c.m. frame with respect to the nonrelativistic spin-one spinor space, we will first determine a basis of linearly independent longitudinal and transverse operators in terms of the only available momenta \vec{k} and \vec{q} and the spin-one operators $S^{[S]}$ ($S = 0, 1, 2$) of rank zero, one and two, using the notation of Fano and Racah [8]. Explicitly, $S^{[0]} = I_3$ is the three-dimensional unit matrix, $S^{[1]}$ the usual spin operator and $S^{[2]}$ is defined by

$$S^{[2]} = [S^{[1]} \times S^{[1]}]^{[2]}. \quad (56)$$

Here, the symbol $[\vec{a} \times \vec{b}]^{[2]}$ stands for the coupling of two vectors \vec{a} and \vec{b} to a second-rank tensor. It is defined by

$$[\vec{a} \times \vec{b}]_{kl}^{[2]} := \frac{1}{2} (a_k b_l + a_l b_k) - \frac{1}{3} \delta_{kl} \vec{a} \cdot \vec{b}. \quad (57)$$

The coupling of a vector \vec{a} with itself will be denoted by the shorthand $a^{[2]} := [\vec{a} \times \vec{a}]^{[2]}$. The longitudinal and transverse operators must have the form

$$\mathcal{O}_L = x, \quad \mathcal{O}_T = \vec{\epsilon} \cdot (\hat{k} \times \vec{y}), \quad (58)$$

where x is a pseudoscalar and \vec{y} a polar vector to be constructed from \hat{k} , \hat{q} and $S^{[S]}$, denoting the unit vector of \vec{v} by $\hat{v} = \vec{v}/|\vec{v}|$. We have four pseudoscalars available

$$\hat{k} \cdot \vec{S}, \quad \hat{q} \cdot \vec{S}, \quad [(\hat{k} \times \hat{q}) \times \hat{k}]^{[2]} \cdot S^{[2]}, \quad \text{and} \quad [(\hat{k} \times \hat{q}) \times \hat{q}]^{[2]} \cdot S^{[2]}. \quad (59)$$

For the vector \vec{y} of the transverse operators one has two types: first \hat{q} times a scalar for which the following types are available

$$1, \quad (\hat{k} \times \hat{q}) \cdot \vec{S}, \quad [\hat{k} \times \hat{q}]^{[2]} \cdot S^{[2]}, \quad \hat{k}^{[2]} \cdot S^{[2]}, \quad \text{and} \quad \hat{q}^{[2]} \cdot S^{[2]}, \quad (60)$$

and second

$$\hat{u} \times \vec{S} \quad \text{and} \quad [\hat{u} \times S^{[2]}]^{[1]}, \quad \text{with} \quad \hat{u} \in \{\hat{k}, \hat{q}\}, \quad (61)$$

where the coupling of a vector with a second-rank tensor to form a vector is given by

$$[\vec{u} \times S^{[2]}]_k^{[1]} = u_l S_{lk}^{[2]}. \quad (62)$$

The resulting thirteen basic operators are listed in Table 6. One should note, that the first four transverse and the first two longitudinal operators correspond to the CGLN-operators for a spin one-half target. The additional operators appear due to the presence of the second rank spin tensor $S^{[2]}$. All other operators, which can be constructed from \vec{k} and \vec{q} and the spin-one operators $S^{[S]}$, can be reduced to linear combinations of the above basic operators with scalar functions of \vec{k}^2 , \vec{q}^2 , and $\vec{k} \cdot \vec{q}$ as coefficients, which in turn may be expressed as functions of the Mandelstam variables.

In complete analogy to the CGLN-amplitudes in e.m. pion production, one can expand the general T -matrix \mathcal{M}_{fi} in terms of the basic operators of Table 6, denoting the longitudinal and transverse parts by a proper superscript

$$\mathcal{M}_{fi}^{L/T} = \chi_{m'}^{[1]\dagger} \left(\sum_{\beta} G_{\beta}^{L/T}(s, t, u) \mathcal{O}_{L/T, \beta} \right) \chi_m^{[1]}, \quad (63)$$

where again $G_{\beta}^{L/T}(s, t, u)$ are invariant functions, and $\beta = 1, \dots, 4$ for the longitudinal and $\beta = 1, \dots, 9$ for the transverse matrix elements. The nonrelativistic spin-one spinors are denoted by $\chi_m^{[1]}$.

In order to relate this representation to the one in (11), we start from the general form of the polarization vector $U(\vec{p}, m)$ of a massive spin-one particle as given in (4). The rest frame polarization three-vector \vec{e}_m is related to the nonrelativistic spin-one spinor $\chi_m^{[1]}$ with z -axis as quantization axis by

$$\vec{e}_m = \vec{P} \chi_m^{[1]}. \quad (64)$$

The components of \vec{P} are (1×3) -matrices

$$P_x = \frac{1}{\sqrt{2}}(-1, 0, 1), \quad P_y = -\frac{i}{\sqrt{2}}(1, 0, 1), \quad P_z = (0, 1, 0), \quad (65)$$

where we note the orthogonality property $P_k P_l^{\dagger} = \delta_{kl}$. It is straightforward to show that the bilinear products of the form $P_k^{\dagger} P_l$, which appear in the basic amplitudes, can be expressed by the spin-one operators of rank zero, one and two according to

$$P_k^{\dagger} P_l = \frac{1}{3} \delta_{kl} I_3 + \frac{i}{2} \varepsilon_{klj} S_j - S_{kl}^{[2]}. \quad (66)$$

Introducing now for convenience the following quantities

$$[\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}), \quad (67)$$

$$N_k = \frac{E_k}{M} - 1 = \frac{(\sqrt{s} - M)^2 + K^2}{2M\sqrt{s}}, \quad (68)$$

$$N_q = \frac{E_q}{M} - 1 = \frac{(\sqrt{s} - M)^2 - m^2}{2M\sqrt{s}}, \quad (69)$$

$$N = \frac{kq}{M^2} - N_k N_q \hat{k} \cdot \hat{q}, \quad (70)$$

$$D_k = \frac{k_0 q}{M} + k N_q \hat{k} \cdot \hat{q}, \quad (71)$$

$$D_q = \frac{q_0 k}{M} + q N_k \hat{k} \cdot \hat{q}, \quad (72)$$

the explicit evaluation then yields for the occurring pseudoscalars

$$S(\epsilon, k, p, q) = -\sqrt{s} [\vec{\epsilon} \vec{k} \vec{q}], \quad (73)$$

$$S(U, k, p, q) = -\sqrt{s} [\vec{k} \vec{q} \vec{P}] \chi_m^{[1]}, \quad (74)$$

$$S(\epsilon, U, k, p) = -\sqrt{s} [\vec{\epsilon} \vec{k} \vec{P}] \chi_m^{[1]}, \quad (75)$$

$$S(U', k, p, q) = -\chi_{m'}^{[1]\dagger} \sqrt{s} [\vec{k} \vec{q} \vec{P}^{\dagger}], \quad (76)$$

$$S(U', \epsilon, k, p) = \chi_{m'}^{[1]\dagger} k \sqrt{s} \left(N_q [\vec{\epsilon} \hat{k} \hat{q}] \hat{q} \cdot \vec{P}^{\dagger} + [\vec{\epsilon} \hat{k} \vec{P}^{\dagger}] \right). \quad (77)$$

Evidently, the pseudoscalars $S(\epsilon, k, p, q)$, $S(\epsilon, U, k, p)$ and $S(U', \epsilon, k, p)$ lead to purely transverse currents. For the scalar products with the initial and final target polarization vectors appearing in Table 2 and 4 one finds

$$U' \cdot U = -\chi_{m'}^{[1]\dagger} \left(\vec{P}^\dagger \cdot \vec{P} + N_k \hat{k} \cdot \vec{P}^\dagger \hat{k} \cdot \vec{P} + N_q \hat{q} \cdot \vec{P}^\dagger \hat{q} \cdot \vec{P} - N \hat{q} \cdot \vec{P}^\dagger \hat{k} \cdot \vec{P} \right) \chi_m^{[1]}, \quad (78)$$

$$U \cdot k = -\frac{\sqrt{s}}{M} \vec{k} \cdot \vec{P} \chi_m^{[1]}, \quad (79)$$

$$U \cdot q = -\left(D_q \hat{k} \cdot \vec{P} + \vec{q} \cdot \vec{P} \right) \chi_m^{[1]}, \quad (80)$$

$$U' \cdot k = -\chi_{m'}^{[1]\dagger} \left(D_k \hat{q} \cdot \vec{P}^\dagger + \vec{k} \cdot \vec{P}^\dagger \right), \quad (81)$$

$$U' \cdot q = -\chi_{m'}^{[1]\dagger} \frac{\sqrt{s}}{M} \vec{q} \cdot \vec{P}^\dagger. \quad (82)$$

Representing now the amplitudes in a reduced operator form by splitting off the spin functions

$$\Omega_\alpha = \chi_{m'}^{[1]\dagger} \tilde{\Omega}_\alpha \chi_m^{[1]}, \quad (83)$$

one finds for the $\tilde{\Omega}_\alpha$ the expressions listed in Table 7, where we have introduced the shorthand

$$\begin{aligned} \Sigma(\vec{a}, \vec{b}) &= \vec{a} \cdot \vec{P}^\dagger \vec{b} \cdot \vec{P} \\ &= \frac{1}{3} \vec{a} \cdot \vec{b} + \frac{i}{2} \vec{S} \cdot (\vec{a} \times \vec{b}) - a_k S_{kl}^{[2]} b_l \\ &= \frac{1}{3} \vec{a} \cdot \vec{b} + \frac{i}{2} \vec{S} \cdot (\vec{a} \times \vec{b}) - [\vec{a} \times \vec{b}]^{[2]} \cdot \vec{S}^{[2]}. \end{aligned} \quad (84)$$

Note, that for the longitudinal operators $\tilde{\Omega}_{10}$ through $\tilde{\Omega}_{13}$ the specific form of the longitudinal polarization vector of the virtual photon of (7) has been used already. As mentioned before, the operators $\tilde{\Omega}_1$ through $\tilde{\Omega}_9$ are purely transverse and, therefore, suffice for describing photoproduction, whereas $\tilde{\Omega}_{10}$ through $\tilde{\Omega}_{13}$ contain only charge and longitudinal current contributions.

It is now straightforward with the help of (84) to expand the operators $\tilde{\Omega}_\alpha$ in terms of the non-relativistic operators. To this end we first note for the various Σ -expressions in Table 7 the following relations to the nonrelativistic transverse and longitudinal operators $\mathcal{O}_{L/T,\beta}$

$$[\vec{e} \hat{k} \hat{q}] \Sigma(\hat{k}, \hat{k}) = \frac{1}{3} \mathcal{O}_{T,1} - \mathcal{O}_{T,5}, \quad (85)$$

$$[\vec{e} \hat{k} \hat{q}] \Sigma(\hat{q}, \hat{q}) = \frac{1}{3} \mathcal{O}_{T,1} - \mathcal{O}_{T,7}, \quad (86)$$

$$[\vec{e} \hat{k} \hat{q}] \Sigma(\hat{k}, \hat{q}) = \frac{1}{3} \hat{k} \cdot \hat{q} \mathcal{O}_{T,1} + \frac{i}{2} \mathcal{O}_{T,2} - \mathcal{O}_{T,6}, \quad (87)$$

$$[\vec{e} \hat{k} \hat{q}] \Sigma(\hat{q}, \hat{k}) = \frac{1}{3} \hat{k} \cdot \hat{q} \mathcal{O}_{T,1} - \frac{i}{2} \mathcal{O}_{T,2} - \mathcal{O}_{T,6}, \quad (88)$$

$$\Sigma(\vec{e} \times \hat{k}, \hat{k}) = \frac{i}{2} \mathcal{O}_{T,3} - \mathcal{O}_{T,8}, \quad (89)$$

$$\Sigma(\hat{k}, \vec{e} \times \hat{k}) = -\frac{i}{2} \mathcal{O}_{T,3} - \mathcal{O}_{T,8}, \quad (90)$$

$$\Sigma(\vec{e} \times \hat{k}, \hat{q}) = \frac{1}{3} \mathcal{O}_{T,1} + \frac{i}{2} \mathcal{O}_{T,4} - \mathcal{O}_{T,9}, \quad (91)$$

$$\Sigma(\hat{q}, \vec{e} \times \hat{k}) = \frac{1}{3} \mathcal{O}_{T,1} - \frac{i}{2} \mathcal{O}_{T,4} - \mathcal{O}_{T,9}, \quad (92)$$

$$\Sigma(\hat{k} \times \hat{q}, \hat{k}) = -\frac{i}{2} \hat{k} \cdot \hat{q} \mathcal{O}_{L,1} + \frac{i}{2} \mathcal{O}_{L,2} - \mathcal{O}_{L,3}, \quad (93)$$

$$\Sigma(\hat{k}, \hat{k} \times \hat{q}) = \frac{i}{2} \hat{k} \cdot \hat{q} \mathcal{O}_{L,1} - \frac{i}{2} \mathcal{O}_{L,2} - \mathcal{O}_{L,3}, \quad (94)$$

$$\Sigma(\hat{k} \times \hat{q}, \hat{q}) = -\frac{i}{2} \mathcal{O}_{L,1} + \frac{i}{2} \hat{k} \cdot \hat{q} \mathcal{O}_{L,2} - \mathcal{O}_{L,4}, \quad (95)$$

$$\Sigma(\hat{q}, \hat{k} \times \hat{q}) = \frac{i}{2} \mathcal{O}_{L,1} - \frac{i}{2} \hat{k} \cdot \hat{q} \mathcal{O}_{L,2} - \mathcal{O}_{L,4}. \quad (96)$$

This then leads to the expansion of the reduced operators $\tilde{\Omega}_\alpha$ by the $\mathcal{O}_{L/T,\beta}$ in the form

$$\tilde{\Omega}_\alpha = \begin{cases} \bar{g}_\alpha^T \sum_{\beta=1}^9 g_{\alpha,\beta}^T \mathcal{O}_{T,\beta} & \text{for } \alpha = 1, \dots, 9, \\ \bar{g}_\alpha^L \sum_{\beta=1}^4 g_{\alpha,\beta}^L \mathcal{O}_{L,\beta} & \text{for } \alpha = 10, \dots, 13, \end{cases} \quad (97)$$

where the coefficients $\bar{g}_\alpha^{T/L}$ and $g_{\alpha,\beta}^{T/L}$ are listed in Tables 8 and 9, respectively. Corresponding relations between the invariant functions $F_\alpha(s, t, u)$ of (11) and $G_\beta^{L/T}(s, t, u)$ of (63) follow

$$G_\beta^L(s, t, u) = \sum_{\alpha=10}^{13} F_\alpha(s, t, u) \bar{g}_\alpha^L g_{\alpha,\beta}^L, \quad (98)$$

$$G_\beta^T(s, t, u) = \sum_{\alpha=1}^9 F_\alpha(s, t, u) \bar{g}_\alpha^T g_{\alpha,\beta}^T. \quad (99)$$

With this we will close the present formal study. It will be a task for the future to derive explicit expressions in terms of the invariant functions for observables like differential cross section, target asymmetries and polarization components of the final target state. Furthermore, one needs to construct the invariant functions for specific reaction models like, for example, coherent pseudoscalar meson production from the deuteron in the impulse approximation. In a later stage, also rescattering and other two-body contributions can be treated accordingly.

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Appendix A: Linear relations for the Levy-Civita tensor

In this appendix we will show, that expressions containing products of three Levi-Civita tensors can be reduced to ones with one Levi-Civita tensor only. The Levi-Civita tensor is defined by

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an even permutation of } \{0, 1, 2, 3\}, \\ -1 & \text{if it is an odd permutation,} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A.1})$$

Products of two Levi-Civita tensors can be reduced to expressions containing only the metric tensor $g^{\mu\nu}$ (see, for example, the Appendix A of [9]). Therefore, among the products of three Levi-Civita tensors, the unconnected ones, i.e., those without contractions between them, reduce to one Levi-Civita tensor and do not lead to a linear relation. For the connected ones, only the following identities for contractions over one and two index pairs are useful for deriving linear relations

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\sigma}{}^{\mu'\nu'\rho'} = \det(g^{\alpha\alpha'}), \quad \alpha = \mu, \nu, \rho, \quad \alpha' = \mu', \nu', \rho', \quad (\text{A.2})$$

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\rho\sigma}{}^{\mu'\nu'} = -2(g^{\mu\mu'} g^{\nu\nu'} - g^{\mu\nu'} g^{\nu\mu'}). \quad (\text{A.3})$$

Among the various possible contractions of three Levi-Civita tensors over one or two index pairs between two factors, the one with two-index contractions in both cases is also trivial

$$\varepsilon^{\mu\nu\rho'\sigma'} \varepsilon_{\rho'\sigma'\mu'\nu'} \varepsilon^{\mu'\nu'\rho\sigma} = -4\varepsilon^{\mu\nu\rho\sigma}. \quad (\text{A.4})$$

Thus only two nontrivial cases remain. The first relation follows from the contraction of three Levi-Civita tensors over one index pair for the first product and two index pairs for the second, i.e. $\varepsilon^{\mu\nu\rho\sigma'} \varepsilon_{\sigma'}{}^{\lambda\nu'\rho'} \varepsilon_{\nu'\rho'}{}^{\sigma\tau}$, resulting in

$$\varepsilon^{\mu\nu\rho\sigma} g^{\tau\lambda} + \varepsilon^{\nu\rho\sigma\tau} g^{\mu\lambda} + \varepsilon^{\rho\sigma\tau\mu} g^{\nu\lambda} + \varepsilon^{\sigma\tau\mu\nu} g^{\rho\lambda} + \varepsilon^{\tau\mu\nu\rho} g^{\sigma\lambda} = 0. \quad (\text{A.5})$$

The second relation follows from the contraction over only one index pair for each product, i.e. $\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\sigma}{}^{\alpha\beta\gamma} \varepsilon_{\gamma}{}^{\delta\lambda\tau}$, yielding

$$\begin{aligned} & \varepsilon^{\mu\nu\rho\delta} (g^{\alpha\lambda} g^{\beta\tau} - g^{\alpha\tau} g^{\beta\lambda}) + \varepsilon^{\mu\nu\rho\lambda} (g^{\alpha\tau} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\tau}) + \varepsilon^{\mu\nu\rho\tau} (g^{\alpha\delta} g^{\beta\lambda} - g^{\alpha\lambda} g^{\beta\delta}) = \\ & \varepsilon^{\mu\delta\lambda\tau} (g^{\alpha\nu} g^{\beta\rho} - g^{\alpha\rho} g^{\beta\nu}) + \varepsilon^{\nu\delta\lambda\tau} (g^{\alpha\rho} g^{\beta\mu} - g^{\alpha\mu} g^{\beta\rho}) + \varepsilon^{\rho\delta\lambda\tau} (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}). \end{aligned} \quad (\text{A.6})$$

By contraction with an appropriate number of Lorentz vectors, the first relation can be cast into the form

$$S(a, b, c, d) e \cdot f + S(b, c, d, e) a \cdot f + S(c, d, e, a) b \cdot f + S(d, e, a, b) c \cdot f + S(e, a, b, c) d \cdot f = 0. \quad (\text{A.7})$$

Note, that this relation is totally antisymmetric in the arguments (a, b, c, d, e) .

Defining for convenience as a shorthand

$$[a, b; c, d] := (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c), \quad (\text{A.8})$$

the second relation becomes correspondingly

$$\begin{aligned} & S(a, b, c, d)[e, f; g, h] + S(a, b, c, e)[f, d; g, h] + S(a, b, c, f)[d, e; g, h] = \\ & S(a, d, e, f)[b, c; g, h] + S(b, d, e, f)[c, a; g, h] + S(c, d, e, f)[a, b; g, h]. \end{aligned} \quad (\text{A.9})$$

This relation is totally antisymmetric in the group of variables (a, b, c) , (d, e, f) , and (g, h) . Furthermore, it is symmetric under the common interchange $(a, b, c) \leftrightarrow (d, e, f)$. This property will be useful in the following application. From these relations follows furthermore, that higher order, odd power products of the Levi-Civita tensor can always be reduced to linear expressions.

Appendix B: Linear relations between invariant amplitudes

We will first exploit the relation in (A.7). In view of its symmetry property, one can distinguish two cases with respect to the three polarization vectors (U', U, ϵ) .

(i) All three belong to the group of variables (a, b, c, d) . In this case one finds

$$\begin{aligned} & S(U', \epsilon, U, x) y \cdot z + U' \cdot z S(\epsilon, U, x, y) - S(U', U, x, y) \epsilon \cdot z \\ & + S(U', \epsilon, x, y) U \cdot z - S(U', \epsilon, U, y) x \cdot z = 0, \end{aligned} \quad (\text{B.1})$$

which results in the following relation between the amplitudes of Table 2

$$\Omega_d(x, y, z) = \Omega_a(x) y \cdot z - \Omega_a(y) x \cdot z + \Omega_b(x, y, z) + \Omega_c(x, y, z). \quad (\text{B.2})$$

It allows to eliminate completely $\Omega_d(x, y, z)$ in favour of Ω_a , Ω_b and Ω_c .

(ii) One of the polarization vectors is identified with f in (A.7). This yields three relations. First setting f equal to U and U' , one obtains

$$\Omega_b(x, y, z) + \Omega_b(z, x, y) + \Omega_b(y, z, x) + [\Omega_e(U', \epsilon, U) - \Omega_e(\epsilon, U', U)] \varepsilon(x, y, z), \quad (\text{B.3})$$

$$\Omega_c(x, y, z) + \Omega_c(z, x, y) + \Omega_c(y, z, x) = [\Omega_e(\epsilon, U', U) - \Omega_e(U, U', \epsilon)] \varepsilon(x, y, z), \quad (\text{B.4})$$

where we have defined

$$\varepsilon(x, y, z) = \begin{cases} +1 & \text{if } \{x, y, z\} \text{ is an even permutation of } \{k, p, q\}, \\ -1 & \text{if it is an odd permutation,} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.5})$$

These two relations serve to eliminate $\Omega_b(p, q, k)$ and $\Omega_c(p, q, k)$.

Finally, the last relation in this group which follows by setting $f = \epsilon$

$$\Omega_d(x, y, z) + \Omega_d(y, z, x) + \Omega_d(z, x, y) = [\Omega_e(U', U, \epsilon) - \Omega_e(U, U', \epsilon)] \varepsilon(x, y, z), \quad (\text{B.6})$$

does not constitute an additional constraint. It is fulfilled identically by use of the previous relations (B.2) through (B.4).

The exploitation of the second relation (A.9) is more involved. One may distinguish five groups according to the location of the polarization vectors among the arguments (a, b, c, d, e, f, g, h) . They are listed in Table B.1.

Before evaluating these various groups, several shorthand notations will be introduced

$$Q_a(x; y, z; u, v) := [y, z; u, v] \Omega_a(x), \quad (\text{B.7})$$

which is antisymmetric in (y, z) and in (u, v) and symmetric under the interchange $(y, z) \leftrightarrow (u, v)$. Furthermore, for the amplitudes of type Ω_α we define for $\alpha \in \{b, c\}$

$$Q_\alpha(x, y, z; u, v) := u \cdot x \Omega_\alpha(y, z, v) + u \cdot y \Omega_\alpha(z, x, v) + u \cdot z \Omega_\alpha(x, y, v), \quad (\text{B.8})$$

and for $\beta \in \{f, g, h\}$

$$Q_\beta(x, y, z; u, v) := \varepsilon(x, y, z) \Omega_\beta(u, v), \quad (\text{B.9})$$

which are both totally antisymmetric in (x, y, z) . And finally we define

$$\mathcal{O}_a(x, y, z; u, v) := Q_a(x; y, z; u, v) + Q_a(y; z, x; u, v) + Q_a(z; x, y; u, v), \quad (\text{B.10})$$

$$\mathcal{O}_\alpha(x, y, z; u, v) := Q_\alpha(x, y, z; u, v) - Q_\alpha(x, y, z; v, u), \quad (\text{B.11})$$

for amplitudes of type $\alpha \in \{b, c, f, g, h\}$, which are again all antisymmetric in (x, y, z) and in (u, v) .

Now we will consider the relations which correspond to the various groups in Tab. B.1.

(1) The first group leads straightforwardly to

$$\mathcal{O}_a(x, y, z; u, v) + \mathcal{O}_h(x, y, z; u, v) = \mathcal{O}_f(x, y, z; u, v) + \mathcal{O}_g(x, y, z; u, v). \quad (\text{B.12})$$

(2) The second group leads to three relations

$$\mathcal{O}_a(x, y, z; u, v) = \mathcal{O}_b(x, y, z; u, v) + \mathcal{O}_g(x, y, z; u, v), \quad (\text{B.13})$$

$$\mathcal{O}_a(x, y, z; u, v) = \mathcal{O}_c(x, y, z; u, v) + \mathcal{O}_f(x, y, z; u, v), \quad (\text{B.14})$$

$$\mathcal{O}_a(x, y, z; u, v) = \mathcal{O}_b(x, y, z; u, v) + \mathcal{O}_c(x, y, z; u, v) + \mathcal{O}_h(x, y, z; u, v). \quad (\text{B.15})$$

From these one can also derive the relation of group (1).

(3) For the third group one finds, using the identity II of Appendix C,

$$Q_f(x, y, z; u, v) - Q_h(x, y, z; u, v) = Q_b(x, y, z; u, v), \quad (\text{B.16})$$

$$Q_g(x, y, z; u, v) - Q_h(x, y, z; u, v) = -Q_c(x, y, z; v, u), \quad (\text{B.17})$$

$$\begin{aligned} Q_g(x, y, z; v, u) - Q_f(x, y, z; u, v) &= -\mathcal{O}_a(x, y, z; u, v) - Q_b(x, y, z; v, u) \\ &\quad - Q_c(x, y, z; v, u). \end{aligned} \quad (\text{B.18})$$

It is easy to see, that these relations contain the foregoing ones of the groups (1) and (2).

(4) For this group one obtains

$$\begin{aligned} Q_f(x, y, z; v, u) - Q_h(x, y, u; v, z) &= Q_b(y, z, u; v, x) - Q_b(x, z, u; v, y) \\ &\quad + Q_h(y, z, u; v, x) - Q_h(x, z, u; v, y), \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} Q_g(x, y, z; u, v) - Q_g(u, z, x; y, v) &= Q_c(u, y, x; v, z) - Q_c(y, z, u; v, x) \\ &\quad + Q_h(y, z, u; x, v) - Q_h(u, y, x; z, v), \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned} Q_g(x, y, z; v, u) - Q_g(x, y, u; v, z) &+ Q_f(x, y, z; z, v) - Q_f(x, y, u; z, v) = \\ &\mathcal{O}_a(x, y, v; z, u) - \mathcal{O}_a(z, u, v; x, y) + Q_b(x, y, u; v, z) \\ &- Q_b(x, y, z; v, u) + Q_c(x, y, u; v, z) - Q_c(x, y, z; v, u). \end{aligned} \quad (\text{B.21})$$

In these relations, originally also Q_e appears formally, but its contribution drops out due to the fact that

$$v \cdot [x\varepsilon(y, z, u) - y\varepsilon(z, u, x) + z\varepsilon(u, x, y) + u\varepsilon(x, y, z)] = 0 \quad (\text{B.22})$$

according to the identity (C.1) in Appendix C. Also these relations are contained in the ones of group (3).

(5) Finally, for the last group one has

$$\mathcal{O}_\alpha(x, y, z; u, v) + \mathcal{O}_\alpha(x, y, u; v, z) + \mathcal{O}_\alpha(x, y, v; z, u) = \mathcal{O}_\alpha(z, u, v; x, y) \quad (\text{B.23})$$

for $\alpha \in \{f, g, h\}$, which are identities because of (C.2). This completes the search for all possible relations among the various amplitudes listed in Table 1.

Thus, of all relations of the various groups, only the ones of group (3) constitute new information, which allow to further eliminate some of the amplitudes Ω_α . To this end we set first in (B.16) and (B.17) $(x, y, z) = (k, p, q)$ and then $(u, v) = (x, y)$ and find

$$\Omega_f(x, y) = \Omega_h(x, y) + Q_b(k, p, q; x, y), \quad (\text{B.24})$$

$$\Omega_g(x, y) = \Omega_h(x, y) - Q_c(k, p, q; y, x), \quad (\text{B.25})$$

by which Ω_f and Ω_g can be eliminated. Inserting these expressions into (B.18) one finds

$$\mathcal{O}_b(k, p, q; x, y) + \mathcal{O}_c(k, p, q; x, y) = \mathcal{O}_a(k, p, q; x, y) - \mathcal{O}_h(k, p, q; x, y), \quad (\text{B.26})$$

which corresponds to (B.15). Because of the asymmetry in (x, y) , one has three different cases which serve to eliminate the remaining amplitudes $\Omega_b(p, q; q)$, $\Omega_c(p, q; q)$ and $\Omega_e(\epsilon, U'U)$. To this end we define for $\alpha \in \{b, c\}$

$$\tilde{Q}_\alpha(x, y) := p \cdot x \Omega_\alpha(q, k, y) + q \cdot x \Omega_\alpha(k, p, y), \quad (\text{B.27})$$

$$\tilde{\mathcal{O}}(x, y) := \mathcal{O}_h(k, p, q; x, y) + \tilde{Q}_b(x, y) - \tilde{Q}_b(y, x) + \tilde{Q}_c(x, y) - \tilde{Q}_c(y, x), \quad (\text{B.28})$$

$$\mathcal{P}(x, y) := \mathcal{O}_a(k, p, q; x, y) - \tilde{\mathcal{O}}(x, y). \quad (\text{B.29})$$

Note that $\tilde{\mathcal{O}}(x, y)$ contains gauge invariant amplitudes only but not $\Omega_a(k)$. Then (B.26) becomes

$$k \cdot x (\Omega_b(p, q, y) + \Omega_c(p, q, y)) - k \cdot y [\Omega_b(p, q, x) + \Omega_c(p, q, y)] = \mathcal{P}(x, y), \quad (\text{B.30})$$

where $\mathcal{P}(x, y)$ does not contain $\Omega_b(p, q; x)$ and $\Omega_c(p, q; x)$. Taking first $(x, y) = (k, p)$, one obtains a relation for $\Omega_c(p, q, q)$

$$k^2 \Omega_c(p, q, q) = k^2 \Omega_c(p, q, k) + k \cdot p [\Omega_b(p, q, k) + \Omega_c(p, q, k)] + \mathcal{P}(k, p). \quad (\text{B.31})$$

Then setting $(x, y) = (k, q)$ yields $\Omega_b(p, q, q)$

$$\begin{aligned} k^2 \Omega_b(p, q, q) &= -k^2 \Omega_c(p, q, q) + k \cdot q [\Omega_b(p, q, k) + \Omega_c(p, q, k)] + \mathcal{P}(k, q), \\ &= (k \cdot q - k \cdot p) \Omega_b(p, q, k) + (k \cdot q - k \cdot p - k^2) \Omega_c(p, q, k) \\ &\quad + \mathcal{P}(k, q) - \mathcal{P}(k, p). \end{aligned} \quad (\text{B.32})$$

Finally for $(x, y) = (p, q)$, using the previous two equations, one gets a relation

$$k \cdot p \mathcal{P}(k, q) - k \cdot q \mathcal{P}(k, p) - k^2 \mathcal{P}(p, q) = 0. \quad (\text{B.33})$$

which allows to eliminate $\Omega_a(k)$. To this end we rewrite (B.33) yielding

$$k \cdot p \mathcal{O}_a(k, q) - k \cdot q \mathcal{O}_a(k, p) - k^2 \mathcal{O}_a(p, q) = k \cdot p \tilde{\mathcal{O}}(k, q) - k \cdot q \tilde{\mathcal{O}}(k, p) - k^2 \tilde{\mathcal{O}}(p, q), \quad (\text{B.34})$$

where the rhs is gauge invariant and does not contain $\Omega_a(k)$. Thus the non-gauge invariant amplitudes $\Omega_a(p)$ and $\Omega_a(q)$ have to drop out also on the lhs, and indeed one finds

$$\begin{aligned} (k^2 p^2 q^2 - k^2 (p \cdot q)^2 - p^2 (k \cdot q)^2 - q^2 (k \cdot p)^2 + 2 k \cdot p k \cdot q p \cdot q) \Omega_a(k) = \\ k \cdot p \tilde{\mathcal{O}}(k, q) - k \cdot q \tilde{\mathcal{O}}(k, p) - k^2 \tilde{\mathcal{O}}(p, q). \end{aligned} \quad (\text{B.35})$$

This is the required relation in order to eliminate one of the 14 gauge invariant amplitudes for which we have chosen $\Omega_a(k)$.

Appendix C: Two identities

In this appendix we note two useful identities.

(I) Given a function $f(x, y, z; u)$ which is totally antisymmetric in (x, y, z) , and for which the arguments refer to only three independent variables, then the following identity holds

$$f(x, y, z; u) - f(y, z, u; x) + f(z, u, x; y) - f(u, x, y; z) = 0, \quad (\text{C.1})$$

the proof of which is straightforward and almost trivial, for example, by assuming (x, y, z) to be independent and then setting $u = x$. Because of the asymmetry, this then holds also for $u = y, z$.

(II) Given a function $g(x, y, z; u, v)$ which is totally antisymmetric in (x, y, z) and in (u, v) , and for which all arguments refer to only three independent variables, then the following identity holds

$$g(x, y, z; u, v) + g(x, y, u; v, z) + g(x, y, v; z, u) = g(z, u, v; x, y), \quad (\text{C.2})$$

Since both sides are totally antisymmetric in (z, u, v) , it is sufficient to assume (z, u, v) as independent. Then (x, y) have to coincide with two of them and then it is again straightforward to show the validity of (C.2).

Table 1: Set of basic types of invariant amplitudes with $x, y, z \in \{k, p, q\}$ and $a, b, c \in \{U', \epsilon, U\}$.

notation	explicit form
$\Omega_a(x)$	$S(U', \epsilon, U, x)$
$\Omega_b(x, y, z)$	$S(U', \epsilon, x, y)U \cdot z$
$\Omega_c(x, y, z)$	$U' \cdot z S(\epsilon, U, x, y)$
$\Omega_d(x, y, z)$	$S(U', U, x, y)\epsilon \cdot z$
$\Omega_e(a, b, c)$	$S(a, k, p, q) b \cdot c$
$\Omega_f(x, y)$	$S(U', k, p, q) \epsilon \cdot x U \cdot y$
$\Omega_g(x, y)$	$U' \cdot x S(U, k, p, q) \epsilon \cdot y$
$\Omega_h(x, y)$	$U' \cdot x S(\epsilon, k, p, q) U \cdot y$

Table 2: Set of independent gauge invariant amplitudes Ω_α .

α	notation	explicit form
1	$\Omega_e(\epsilon, U', U)$	$S(\epsilon, k, p, q)U' \cdot U$
2	$\Omega_h(k, k)$	$U' \cdot k S(\epsilon, k, p, q)U \cdot k$
3	$\Omega_h(k, q)$	$U' \cdot k S(\epsilon, k, p, q)U \cdot q$
4	$\Omega_h(q, k)$	$U' \cdot q S(\epsilon, k, p, q)U \cdot k$
5	$\Omega_h(q, q)$	$U' \cdot q S(\epsilon, k, p, q)U \cdot q$
6	$\Omega_b(k, p, k)$	$S(U', \epsilon, k, p)U \cdot k$
7	$\Omega_b(k, p, q)$	$S(U', \epsilon, k, p)U \cdot q$
8	$\Omega_c(k, p, k)$	$U' \cdot k S(\epsilon, U, k, p)$
9	$\Omega_c(k, p, q)$	$U' \cdot q S(\epsilon, U, k, p)$
10	$\Omega_b(k, q, k)$	$S(U', \epsilon, k, q)U \cdot k$
11	$\Omega_b(k, q, q)$	$S(U', \epsilon, k, q)U \cdot q$
12	$\Omega_c(k, q, k)$	$U' \cdot k S(\epsilon, U, k, q)$
13	$\Omega_c(k, q, q)$	$U' \cdot q S(\epsilon, U, k, q)$

Table 3: Set of independent non-gauge invariant amplitudes.

α	notation	explicit form
14	$\Omega_a(p)$	$S(U', \epsilon, U, p)$
15	$\Omega_a(q)$	$S(U', \epsilon, U, q)$
16	$\Omega_e(U', \epsilon, U)$	$S(U', k, p, q) \epsilon \cdot U$
17	$\Omega_e(U, U', \epsilon)$	$U' \cdot \epsilon S(U, k, p, q)$

Table 4: Equivalent set of independent gauge invariant longitudinal amplitudes for electroproduction.

α	notation	explicit form
10	$k \cdot p \Omega_f(k, k) - k^2 \Omega_f(p, k)$	$S(U', k, p, q) U \cdot k (k \cdot p k \cdot \epsilon - k^2 p \cdot \epsilon)$
11	$k \cdot p \Omega_f(k, q) - k^2 \Omega_f(p, q)$	$S(U', k, p, q) U \cdot q (k \cdot p k \cdot \epsilon - k^2 p \cdot \epsilon)$
12	$k \cdot p \Omega_g(k, k) - k^2 \Omega_g(k, p)$	$U' \cdot k S(U, k, p, q) (k \cdot p k \cdot \epsilon - k^2 p \cdot \epsilon)$
13	$k \cdot p \Omega_g(q, k) - k^2 \Omega_g(q, p)$	$U' \cdot q S(U, k, p, q) (k \cdot p k \cdot \epsilon - k^2 p \cdot \epsilon)$

Table 5: Helicity representation of invariant amplitudes.

α	$\Omega_{\alpha, \lambda' \lambda_\gamma \lambda}$
1	$\frac{i}{M^2} \lambda_\gamma k q \sqrt{s} \left[p' p \delta_{\lambda'0} \delta_{\lambda0} - E'(\lambda') E(\lambda) d_{\lambda\lambda'}^1(\theta) \right] d_{-\lambda_\gamma 0}^1(\theta)$
2	$\frac{i}{M^2} \lambda_\gamma k^2 q s \left[q k_0 \delta_{\lambda'0} + k E'(\lambda') d_{0\lambda'}^1(\theta) \right] d_{-\lambda_\gamma 0}^1(\theta) \delta_{\lambda0}$
3	$\frac{i}{M^2} \lambda_\gamma k q \sqrt{s} \left[q k_0 \delta_{\lambda'0} + k E'(\lambda') d_{0\lambda'}^1(\theta) \right] \left[k q_0 \delta_{\lambda0} + q E(\lambda) d_{\lambda0}^1(\theta) \right] d_{-\lambda_\gamma 0}^1(\theta)$
4	$\frac{i}{M^2} \lambda_\gamma k^2 q^2 s^{3/2} \delta_{\lambda'0} \delta_{\lambda0} d_{-\lambda_\gamma 0}^1(\theta)$
5	$\frac{i}{M^2} \lambda_\gamma k q^2 s \delta_{\lambda'0} \left[k q_0 \delta_{\lambda0} + q E(\lambda) d_{\lambda0}^1(\theta) \right] d_{-\lambda_\gamma 0}^1(\theta)$
6	$-\frac{i}{M^2} (-)^{\lambda'} \lambda_\gamma E'(\lambda') k^2 s \delta_{\lambda0} d_{\lambda_\gamma - \lambda'}^1(\theta)$
7	$-\frac{i}{M^2} (-)^{\lambda'} \lambda_\gamma E'(\lambda') k \sqrt{s} \left[k q_0 \delta_{\lambda0} + q E(\lambda) d_{\lambda0}^1(\theta) \right] d_{\lambda_\gamma - \lambda'}^1(\theta)$
8	$\frac{i}{M^2} \lambda E(\lambda) k \sqrt{s} \left[q k_0 \delta_{\lambda'0} + k E'(\lambda') d_{0\lambda'}^1(\theta) \right] \delta_{\lambda\lambda_\gamma}$
9	$\frac{i}{M^2} \lambda E(\lambda) k q s \delta_{\lambda'0} \delta_{\lambda_\gamma \lambda}$
10	$\frac{i}{M} \lambda' \sqrt{K^2} k^3 q s^{3/2} d_{\lambda'0}^1(\theta) \delta_{\lambda_\gamma 0} \delta_{\lambda0}$
11	$\frac{i}{M} \lambda' \sqrt{K^2} k^2 q s d_{\lambda'0}^1(\theta) \delta_{\lambda_\gamma 0} \left[k q_0 \delta_{\lambda0} + q E(\lambda) d_{\lambda0}^1(\theta) \right]$
12	$-\frac{i}{M} \lambda \sqrt{K^2} k^2 q s \left[q k_0 \delta_{\lambda'0} + k E'(\lambda') d_{0\lambda'}^1(\theta) \right] \delta_{\lambda_\gamma 0} d_{\lambda0}^1(\theta)$
13	$-\frac{i}{M} \lambda \sqrt{K^2} k^2 q^2 s^{3/2} \delta_{\lambda'0} \delta_{\lambda_\gamma 0} d_{\lambda0}^1(\theta)$

Table 6: Basic set of independent, nonrelativistic transverse and longitudinal operators.

β	$\mathcal{O}_{T,\beta}$	$\mathcal{O}_{L,\beta}$
1	$\vec{\epsilon} \cdot (\hat{k} \times \hat{q})$	$\hat{k} \cdot \vec{S}$
2	$\vec{\epsilon} \cdot (\hat{k} \times \hat{q}) (\hat{k} \times \hat{q}) \cdot \vec{S}$	$\hat{q} \cdot \vec{S}$
3	$\vec{\epsilon} \cdot (\hat{k} \times (\hat{k} \times \vec{S}))$	$[(\hat{k} \times \hat{q}) \times \hat{k}]^{[2]} \cdot S^{[2]}$
4	$\vec{\epsilon} \cdot (\hat{k} \times (\hat{q} \times \vec{S}))$	$[(\hat{k} \times \hat{q}) \times \hat{q}]^{[2]} \cdot S^{[2]}$
5	$\vec{\epsilon} \cdot (\hat{k} \times \hat{q}) \hat{k}^{[2]} \cdot S^{[2]}$	
6	$\vec{\epsilon} \cdot (\hat{k} \times \hat{q}) [\hat{k} \times \hat{q}]^{[2]} \cdot S^{[2]}$	
7	$\vec{\epsilon} \cdot (\hat{k} \times \hat{q}) \hat{q}^{[2]} \cdot S^{[2]}$	
8	$\vec{\epsilon} \cdot (\hat{k} \times [\hat{k} \times S^{[2]}]^{[1]})$	
9	$\vec{\epsilon} \cdot (\hat{k} \times [\hat{q} \times S^{[2]}]^{[1]})$	

Table 7: Reduced gauge invariant amplitudes $\tilde{\Omega}_\alpha$.

$\tilde{\Omega}_1$	$kq\sqrt{s} [\vec{\epsilon} \hat{k} \hat{q}] \left[1 + N_k \Sigma(\hat{k}, \hat{k}) + N_q \Sigma(\hat{q}, \hat{q}) - N \Sigma(\hat{q}, \hat{k}) \right]$
$\tilde{\Omega}_2$	$-\frac{1}{M} k^2 qs [\vec{\epsilon} \hat{k} \hat{q}] \left[k \Sigma(\hat{k}, \hat{k}) + D_k \Sigma(\hat{q}, \hat{k}) \right]$
$\tilde{\Omega}_3$	$-kq\sqrt{s} [\vec{\epsilon} \hat{k} \hat{q}] \left[kq \Sigma(\hat{k}, \hat{q}) + qD_k \Sigma(\hat{q}, \hat{q}) + kD_q \Sigma(\hat{k}, \hat{k}) + D_k D_q \Sigma(\hat{q}, \hat{k}) \right]$
$\tilde{\Omega}_4$	$-\frac{1}{M^2} k^2 q^2 s^{3/2} [\vec{\epsilon} \hat{k} \hat{q}] \Sigma(\hat{q}, \hat{k})$
$\tilde{\Omega}_5$	$-\frac{1}{M} kq^2 s [\vec{\epsilon} \hat{k} \hat{q}] \left[q \Sigma(\hat{q}, \hat{q}) + D_q \Sigma(\hat{q}, \hat{k}) \right]$
$\tilde{\Omega}_6$	$-\frac{1}{M} k^2 s \left[\Sigma(\vec{\epsilon} \times \hat{k}, \hat{k}) + N_q [\vec{\epsilon} \hat{k} \hat{q}] \Sigma(\hat{q}, \hat{k}) \right]$
$\tilde{\Omega}_7$	$-k\sqrt{s} \left[q \Sigma(\vec{\epsilon} \times \hat{k}, \hat{q}) + D_q \Sigma(\vec{\epsilon} \times \hat{k}, \hat{k}) + N_q [\vec{\epsilon} \hat{k} \hat{q}] \left(q \Sigma(\hat{q}, \hat{q}) + D_q \Sigma(\hat{q}, \hat{k}) \right) \right]$
$\tilde{\Omega}_8$	$k\sqrt{s} \left[k \Sigma(\hat{k}, \vec{\epsilon} \times \hat{k}) + D_k \Sigma(\hat{q}, \vec{\epsilon} \times \hat{k}) \right]$
$\tilde{\Omega}_9$	$\frac{1}{M} kqs \Sigma(\hat{q}, \vec{\epsilon} \times \hat{k})$
$\tilde{\Omega}_{10}$	$\frac{1}{M} \sqrt{K^2} k^3 qs^{3/2} \Sigma(\hat{k} \times \hat{q}, \hat{k})$
$\tilde{\Omega}_{11}$	$\sqrt{K^2} k^2 qs \left[q \Sigma(\hat{k} \times \hat{q}, \hat{q}) + D_q \Sigma(\hat{k} \times \hat{q}, \hat{k}) \right]$
$\tilde{\Omega}_{12}$	$\sqrt{K^2} k^2 qs \left[k \Sigma(\hat{k}, \hat{k} \times \hat{q}) + D_k \Sigma(\hat{q}, \hat{k} \times \hat{q}) \right]$
$\tilde{\Omega}_{13}$	$\frac{1}{M} \sqrt{K^2} k^2 q^2 s^{3/2} \Sigma(\hat{q}, \hat{k} \times \hat{q})$

Table 8: Coefficients for the transformation of the reduced transverse operators $\tilde{\Omega}_\alpha$ ($\alpha = 1, \dots, 9$) to the nonrelativistic operators $\mathcal{O}_{T,\beta}$ according to (97).

α	\bar{g}_α^T	$g_{\alpha,1}^T$	$g_{\alpha,2}^T$	$g_{\alpha,3}^T$	$g_{\alpha,4}^T$
1	$kq\sqrt{s}$	$1 + \frac{1}{3}(N_k + N_q - N\hat{k} \cdot \hat{q})$	$\frac{i}{2}N$		
2	$\frac{1}{M}k^2qs$	$-\frac{1}{3}(k + D_k\hat{k} \cdot \hat{q})$	$\frac{i}{2}D_k$		
3	$kq\sqrt{s}$	$-\frac{1}{3}(kD_q + qD_k + (kq + D_kD_q)\hat{k} \cdot \hat{q})$	$-\frac{i}{2}(kq - D_kD_q)$		
4	$\frac{1}{M^2}k^2q^2s^{3/2}$	$-\frac{1}{3}\hat{k} \cdot \hat{q}$	$\frac{i}{2}$		
5	$\frac{1}{M}kq^2s$	$-\frac{1}{3}(q + D_q\hat{k} \cdot \hat{q})$	$\frac{i}{2}D_q$		
6	$\frac{1}{M}k^2s$	$-\frac{1}{3}N_q\hat{k} \cdot \hat{q}$	$\frac{i}{2}N_q$		
7	$k\sqrt{s}$	$-\frac{1}{3}(q + N_q(q + D_q\hat{k} \cdot \hat{q}))$	$\frac{i}{2}D_qN_q$	$-\frac{i}{2}D_q$	$-\frac{i}{2}q$
8	$k\sqrt{s}$	$\frac{1}{3}D_k$		$-\frac{i}{2}k$	$-\frac{i}{2}D_k$
9	$\frac{1}{M}kqs$	$\frac{1}{3}$	$-\frac{i}{2}$		
α	$g_{\alpha,5}^T$	$g_{\alpha,6}^T$	$g_{\alpha,7}^T$	$g_{\alpha,8}^T$	$g_{\alpha,9}^T$
1	$-N_k$	N	$-N_q$		
2	k	D_k			
3	kD_q	$kq + D_kD_q$	qD_k		
4		1			
5		D_q	q		
6		N_q	$-\frac{i}{2}$	1	
7		D_qN_q	qN_q	D_q	q
8				$-k$	$-D_k$
9					-1

Table 9: Coefficients for the transformation of the reduced longitudinal operators $\tilde{\Omega}_\alpha$ ($\alpha = 10, \dots, 13$) to the nonrelativistic operators $\mathcal{O}_{L,\beta}$ according to (97).

α	\bar{g}_α^L	$g_{\alpha,1}^L$	$g_{\alpha,2}^L$	$g_{\alpha,3}^L$	$g_{\alpha,4}^L$
10	$\frac{1}{M}\sqrt{K^2}k^3qs^{3/2}$	$-\frac{i}{2}\hat{k} \cdot \hat{q}$	$\frac{i}{2}$	-1	
11	$\sqrt{K^2}k^2qs$	$-\frac{i}{2}(q + D_q\hat{k} \cdot \hat{q})$	$\frac{i}{2}(q\hat{k} \cdot \hat{q} + D_q)$	$-D_q$	$-q$
12	$\sqrt{K^2}k^2qs$	$\frac{i}{2}(k\hat{k} \cdot \hat{q} + D_k)$	$-\frac{i}{2}(k + D_k\hat{k} \cdot \hat{q})$	$-k$	$-D_k$
13	$\frac{1}{M}\sqrt{K^2}k^2q^2s^{3/2}$	$\frac{i}{2}$	$-\frac{i}{2}\hat{k} \cdot \hat{q}$		-1

Table B.1: Listing of various groups of relations following from (A.9).

	a, b, c	d, e, f	g, h
(1)	U', ϵ, U		
(2)	U', ϵ ϵ, U U', U	U U' ϵ	
(3)	U', ϵ ϵ, U U', U		U U' ϵ
(4)	U' ϵ U	ϵ U U'	U U' ϵ
(5)	U' ϵ U		ϵ, U U', U U', ϵ